In this problem, P is the transition matrix for an irreducible Markov process with period d. We are asked to find the number of communicating classes and periods for P^k.

First note the following

Lemma 0.1. *Let $d = HCF\{a_1, a_2, \dots, a_n\}$, where the a_i are positive integers. Write*

$$d = b_1 a_1 + \cdots b_n a_n$$

for $b_i \in \mathbb{Z}$. Put $B = \sup_i\{|b_i|\}$ and let $A = \sum_i a_i$. Then for all $m > A^2 B$, we can write

$$md = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$$

with $c_i \in \mathbb{Z}$, $c_i \geq 0$.

Proof. Suppose $m > A^2 B/d$. Write

$$m = qA + r$$

with $q \geq AB/d$ and $0 \leq r < A$. Then

$$md = qAd + rd = qd(\sum_i a_i) + r\sum_i b_i a_i = \sum_i (qd + rb_i)a_i.$$

But $qd \geq AB \geq r|b_i|$ for each i. Hence, $c_i := qd + rb_i \geq 0$ for each i. □

Let I be the state space for the Markov process. Recall that the period is the same for all $x \in$ and it is by definition

$$d = HCF\{a \mid p_{xx}^{(a)} > 0\}.$$

As we go over an increasing sequence of finite subsets of elements in $\{a \mid p_{xx}^{(a)} > 0\}$, the HCF of these subsets get smaller and smaller. Hence, d is actually the HCF of a finite set of a_i in the set. Thus, with notation as in the lemma, for $m > A^2 B/d$,

$$p_{xx}^{(md)} \geq (p^{(a_1)})^{c_1} (p^{(a_2)})^{c_2} \cdots (p^{(a_n)})^{c_n} > 0.$$

For $n \in \mathbb{Z}$, denote by $[n]$ its class in $\mathbb{Z}/d$. For $r \in \mathbb{Z}/d$, we denote by r' an element of $\mathbb{Z}$ such that $[r'] = r$. For $x \in I$ and $r \in \mathbb{Z}/d$, let

$$I_r(x) = \{y \in I \mid \exists n,\ p_{xy}^{(n)} > 0,\ n \equiv r \mod d\}.$$

From the definition, it's clear that if $y \in I_r(x)$ and $z \in I_s(y)$, then $z \in I_{r+s}(x)$.

Proposition 0.2. *Suppose $y \in I_r(x)$. Then $x \in I_{-r}(y)$.*

Proof. We have $p_{yx}^{(m)} > 0$ for some m by irreducibility. Thus, for some n such that $[n] = r$, we have $p_{xx}^{(n+m)} > 0$. But then, $d|(n+m)$, and hence, $[m] = -r$. □

Proposition 0.3. *If $y, z \in I_r(x)$, then $z \in I_0(y)$.*

Proof. Since $x \in I_{-r}(y)$ and $z \in I_r(x)$, we have $z \in I_{-r+r}(y) = I_0(y)$. □

Proposition 0.4. *For $r \neq s \in \mathbb{Z}/d$, $I_r(x) \cap I_s(x) = \phi$.*

Proof. Let $y \in I_r(x) \cap I_s(x)$. Then $x \in I_{-s}(y)$. So $x \in I_{r-s}(x)$. Thus for some $n \in \mathbb{Z}$ such that $[n] = r - s$, we have $p_{xx}^{(n)} > 0$. But then, $d|n$. So $r - s = 0$. □

By irreducibility, $I = \cup_r I_r(x)$, and we have just shown this is a disjoint union.

Proposition 0.5. *Let $y \in I_r(x)$. The for all m sufficiently large such that $m \equiv r \mod d$, we have*

$$p_{xy}^{(m)} > 0.$$

Proof. We have $p_{xy}^{(n)} > 0$ for some $n \equiv r \mod d$. Let $m \equiv r \mod d$ satisfy $m > A^2B + n$ with notation as in the previous lemma. Then $(m-n)/d > A^2B/d$. Hence, $p_{xx}^{(m-n)} > 0$ and

$$p_{xy}^{(m)} \geq p_{xx}^{(m-n)} p_{xy}^{(n)} > 0.$$

□

In particular, if $y \in I_0(x)$, then $p_{xy}^{(n)} > 0$ for all $n \in d\mathbb{Z}$, sufficiently large. This implies that $p_{xy}^{(kn)} > 0$ for $n \in d\mathbb{Z}$ sufficiently large. So all $I_0(x)$ are all in the same communicating class for P^k. Similarly, since $I_j(x) = I_0(y)$ for any $y \in I_j(x)$, we see that all of $I_j(x)$ is in the same communicating class for any j.

Let $g = HCF\{k, d\}$.

Proposition 0.6. *$I_r(x)$ and $I_s(x)$ are in the same communicating class for P^k if and only if $r \equiv s \mod g$.*

Proof. Suppose they are in the same communicating class. Then there are $y \in I_r(x)$ and $z \in I_s(x)$ such that $p_{yz}^{(mk)} > 0$ for some m. So then $z \in I_{[mk]}(y)$, and hence, $z \in I_{r+[mk]}(x)$. This implies that $r + [mk] = s$. Hence, $r' - s' = -mk + ad$ for some a, and hence, $r \equiv s \mod g$.

Conversely, suppose $r \equiv s \mod g$ and let $y \in I_r(x)$, $z \in I_s(x)$. Then

$$r' = s' + ag$$

for some a. Thus, we have $y \in I_{[ag]}(z)$. We can write $ag = ld + nk$. So $y \in I_{[nk]}(z)$. Hence, for all m sufficiently large such that $m \equiv nk \mod d$, we have $p_{zy}^{(m)} > 0$. In particular, we can choose m of the form $m = nk + hdk$ for h large. But then, $p_{zy}^{((n+hd)k)} > 0$. We can switch the role of z and y in this argument, which allows us to conclude that z and y are in the same communicating class for P^k. □

We conclude that there are g communicating classes. Now, I leave it to you to check that the period of P^k is d/g.