Homotopy theory and Diophantine geometry

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For $a, b, c \in \mathbb{Z}$, positive and $n \geq 4$:

$$ax^n + by^n = c \tag{2}$$

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There are *geometric methods* for producing and organizing such solutions:

the method of sweeping lines

and

the chord-tangent method

Rational points on real algebraic curves

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That is, visualize the points

$$P = (\frac{-238740}{238741}, \frac{691}{238741})$$

and

$$Q = (\frac{2340922881}{58675600}, \frac{113259286337279}{449455096000})$$

on the real algebraic curves $x^2 + y^2 = 1$ and $y^2 = x^3 - 2$.

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However, real picture is not very useful for equation (2).

The difficulty is due in part to the *topology of the complex solution set*.

Topology of complex curves

That is, for a polynomial f, if we are interested in

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we are obliged to examine the topology of

$$X_f(\mathbb{C}) := \{(z,w) \in \mathbb{C}^2 : f(z,w) = 0\}$$

and its closure

$$\overline{X_f(\mathbb{C})}$$

in projective space $\mathbb{P}^2(\mathbb{C})$.

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We use this to speak of the *genus* of the equation f.

In the previous examples,
equation (0) has genus 0;
equation (1) has genus 1;
equation (2) has genus 2.

Topology of complex curves and the theorem of Faltings

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Most equations of degree ≥ 4, e.g.,

$$ax^n + by^n = c \qquad (n \geq 4),$$

have genus ≥ 2.

The effective Mordell problem

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Many years of experience indicates that the real picture is not very useful for this problem.

Non-abelian fundamental groups

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This non-abelian nature is strongly linked to the difficulty of the rational solution set, but may provide a natural arena for resolving the effective Mordell problem.

Naturally linked to a *p-adic embedding* that can be used to analyze the arithmetic structure.

Interlude: *L*-functions and elliptic curves

The zero set of an equation of genus one is called an *elliptic curve*, usually written in the (Weierstrass minimal) form

$$E: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

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Still many unresolved problems in this direction, but a *conjectural picture* is relatively complete. In particular, we can always compute $E(\mathbb{Q})$ in practice, given enough time.

Interlude: L-functions and elliptic curves

It is a theorem of Wiles (and Taylor and Breuil, Conrad, Diamond, Taylor) that there is an arithmetic uniformization

$$\pi : \mathbb{H} \longrightarrow E(\mathbb{C}),$$

where $\mathbb{H}$ is the complex upper-half plane:

$$\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}.$$

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'Arithmetic' here refers to the fact that the fibers of π should contain the orbits of a congruence subgroup

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \mod N\}$$

of $SL_2(\mathbb{Z})$.

Interlude: *L*-functions and elliptic curves

The *L-function* of E is the entire function

$$L(E,s) := \frac{1}{\Gamma(s)} \int_0^\infty \pi^*(\alpha)(iy)^s \frac{dy}{y},$$

where α is given by the formula:

$$\alpha := \frac{dx}{2y + a_1 x + a_3}.$$

Interlude: The Coates-Wiles theorem

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Theorem (Coates-Wiles)

If $L(E_d, 1) \neq 0$, then $E(\mathbb{Q})$ is finite

For example, for E_2, $L(E_2, 1) = 0$.

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- Special case of a theorem on *elliptic curves with complex multiplication*.
- Has been generalized to non-CM elliptic curves over $\mathbb{Q}$ by Kolyvagin, Rubin, and Kato.
- The theorem relies on a study of the *homology* of the elliptic curve.
- Would like to generalize the ideas to other varieties, for example, curves of higher genus using *homotopy theory*.

Homotopy theory: Points and principal bundles

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Given an equation

$$f(x, y) = 0$$

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The *fundamental groupoid* is made up of the path spaces

$$\pi_1(X(\mathbb{C}); a, b)$$

as the two points a and b vary over $X(\mathbb{C})$, together with the composition

$$\pi_1(X(\mathbb{C}); b, c) \times \pi_1(X(\mathbb{C}); a, b) \to \pi_1(X(\mathbb{C}); a, c)$$

obtained by concatenating paths.

Homotopy theory: Points and principal bundles

The portion that originates at a fixed base-point b is comprised of the fundamental group

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We will focus mostly on the category of *principal bundles* for the group $\pi_1(X(\mathbb{C}), b)$ made up by the path spaces $\pi_1(X(\mathbb{C}); b, x)$.

Homotopy theory: Points and principal bundles

This means that there is a group action

$$\pi_1(X(\mathbb{C}); b, x) \times \pi_1(X(\mathbb{C}), b) \longrightarrow \pi_1(X(\mathbb{C}); b, x)$$

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Alternatively, any choice of a path $p \in \pi_1(X(\mathbb{C}); b, x)$ determines a bijection

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The proposal of arithmetic homotopy is to encode solutions to the equation, i.e., points on X, into the structures $\pi_1(X(\mathbb{C}); b, x)$.

Arithmetic homotopy and Diophantine geometry

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Standard linearization: the group ring

$$\mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)].$$

Obtain thereby, a number of additional structures.

Arithmetic homotopy and Diophantine geometry

The group ring is a Hopf algebra with comultiplication

$$\Delta : \mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)] \rightarrow \mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)] \otimes \mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)]$$

determined by the formula

$$\Delta(g) = g \otimes g$$

for $g \in \pi_1(X(\mathbb{C}), b)$.

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for $g \in \pi_1(X(\mathbb{C}), b)$.
Inside the group ring there is the augmentation ideal

$$J \subset \mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)]$$

generated by elements of the form $g - 1$.

Arithmetic homotopy and Diophantine geometry

Completion:

$$A = \mathbb{Q}_p[[\pi_1(X(\mathbb{C}), b)]] := \varprojlim_n \mathbb{Q}_p[\pi_1(X(\mathbb{C}), b)]/J^n,$$

whose elements can be thought of as non-commuative formal power series in elements $g - 1$, $g \in \pi_1$.

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The previous co-product carries over to an algebra homomorphism

$$\Delta : A \longrightarrow A \hat{\otimes} A := \varprojlim A/J^n \otimes A/J^m,$$

turning A into a *complete Hopf algebra*.
Study of such structures originates in *rational homotopy theory*, with which we are actually concerned from a motivic point of view.

Arithmetic homotopy and Diophantine geometry

One defines the 'group-like elements' in the complete Hopf algebra:

$$U = \{g \in A \mid \Delta(g) = g \otimes g\}.$$

The elements of the discrete fundamental group give rise to elements of U, but there are many more. For example, given $g \in \pi_1$, one can obtain elements of U using $\mathbb{Q}_p$-powers of g:

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$$g^\lambda := \exp(\lambda \log(g)).$$

U can be thought of as a group generated by such $\mathbb{Q}_p$-powers.

The group U is in fact very large, with the structure of an infinite-dimensional pro-algebraic group over $\mathbb{Q}_p$, the $\mathbb{Q}_p$-*pro-unipotent completion* of $\pi_1(X(\mathbb{C}), b)$:

$$\pi_1(X(\mathbb{C}), b) \longrightarrow U.$$

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$$\pi_1(X(\mathbb{C}), b) \longrightarrow U.$$

The principal bundles of paths can be completed as well, to give

$$P(x) := \pi_1(X(\mathbb{C}), b)\backslash[\pi_1(X(\mathbb{C}); b, c) \times U],$$

which are principal bundles for U.

Arithmetic homotopy and Diophantine geometry

Now, when b and x are both rational points both U and $P(x)$ admit a very large group of hidden symmetries, a continuous action of

$$G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}).$$

The symmetry arises from a reinterpretation of these constructions in terms of the étale topology of the scheme X.

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The principal bundles $P(x)$ themselves are non-trivial for the *étale topology* of $\mathrm{Spec}(\mathbb{Q})$.

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-If p is chosen large enough and the fundamental group is non-abelian, then the structure $P(x)$ completely determines the point x. That is, if

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-Can classify such structures, using a pro-algebraic moduli space

$$H^1_f(G, U),$$

of principal bundles described using non-abelian continuous group cohomology: The $\mathbb{Q}_p$-*Selmer scheme* of X.

Arithmetic homotopy and Diophantine geometry

Arithmetic homotopy and Diophantine geometry

Each $P(x)$ determines an element of this space.

$$X(\mathbb{Q}) \longrightarrow H^1_f(G, U);$$

$$x \mapsto [P(x)];$$

defining an embedding of the rational points of X into a *moduli space of principal bundles.*

Thereby, the $\mathbb{Q}$-points $X(\mathbb{Q})$ become embedded in a canonical algebraic family.

Arithmetic homotopy and Diophantine geometry

Localization:

$$\begin{array}{ccc} X(\mathbb{Q}) & \longrightarrow & X(\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ H^1_f(G, U) & \xrightarrow{\mathrm{loc}_p} & H^1_f(G_p, U) \end{array}$$

Here $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \subset G$.

Arithmetic homotopy and Diophantine geometry

The local moduli space and the classifying map can be computed using non-abelian p-adic Hodge theory.

$$H^1_f(G_p, U) \simeq \mathbb{Q}_p^\infty$$

and the map

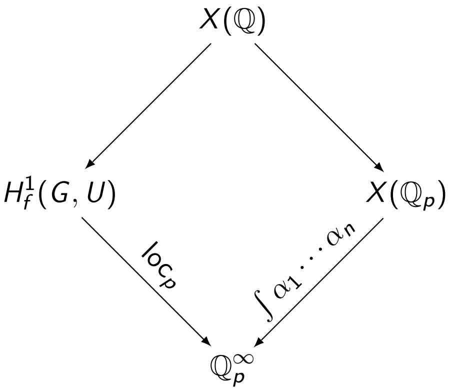
$$X(\mathbb{Q}_p) \longrightarrow H^1_f(G_p, U)$$

has coordinates that are p-adic iterated integrals:

$$z \mapsto (\cdots, \int_b^z \alpha_1\alpha_2\cdots\alpha_n, \cdots).$$

Arithmetic homotopy and Diophantine finiteness

These constructions are captured by the following diagram:



Arithmetic homotopy and Diophantine finiteness

so that

$$Im(H^1_f(G,U)) \subset \mathbb{Q}_p^\infty$$

is an algebraic subspace such that

$$X(\mathbb{Q}) \subset Im(H^1_f(G,U)) \cap X(\mathbb{Q}_p) \subset \mathbb{Q}_p^\infty$$

Arithmetic homotopy and Diophantine finiteness

For the equation

$$ax^n + by^n = c$$

($n \geq 4$), one can show (joint work with John Coates) that

$$Im(H^1_f(G, U)) \cap X(\mathbb{Q}_p) \subset \mathbb{Q}_p^\infty$$

is finite, and deduce from this the finiteness of points.

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This sparseness in turn implies that the closure of

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is a proper algebraic subspace.
The finiteness of the intersection follows then from the analytic fact that

$$Im(X(\mathbb{Q}_p)) \subset \mathbb{Q}_p^{\infty}$$

is a *space-filling curve*, which therefore must meet a proper subspace in a discrete set of points.

Arithmetic homotopy and Diophantine finiteness

But it is further plausible that

$$Im(H^1_f(G, U)) \cap X(\mathbb{Q}_p) \subset \mathbb{Q}_p^\infty$$

is *computable*, since

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is algebraic.

The intersection might be regarded as a 'computable approximation' to $X(\mathbb{Q})$.

The effective Mordell problem

A tool for the effective Mordell problem:

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Appears to require further input from Grothendieck's *anabelian geometry*.