Problem Sheet 4

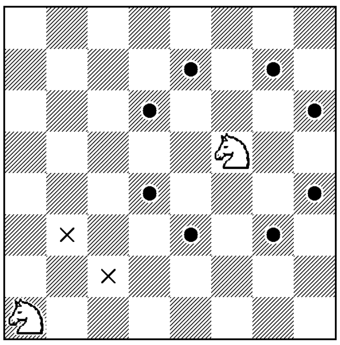
1. Find the stationary distributions of the following transition matrices. In each case describe the limiting behaviour of $p_{12}^{(n)}$ as $n \to \infty$.

$$\text{(i)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. A fair die is thrown repeatedly. Let X_n denote the sum of the first n throws. Find

$$\lim_{n\to\infty} \mathbb{P}\left(X_n \text{ is a multiple of } 11\right).$$

3. A knight performs a random walk on a chessboard, making each possible move with equal probability at each step. If the knight starts in the bottom left corner, how many moves on average will it take to return there? (The knight's possible moves from two different positions are shown in the picture.) [*Hint: consider the "random walk on a graph" from lectures, in which the equilibrium probability of a vertex is proportional to its number of neighbours.*]



4. A frog jumps on an infinite ladder. At each jump, he goes up one step with probability $1 - p$, and falls all the way to the bottom with probability p.

   Represent his position on the ladder as a Markov chain, and find its equilibrium distribution.

   If the frog has just fallen to the bottom, on average how many jumps will it take before he next reaches step k? (One approach: consider the mean return time from k to itself.)

5. Starting from some fixed time, requests at a web server arrive at an average rate of 2 per second, according to a Poisson process. Find: (a) the probability that the first request arrives within 2 seconds. (b) the distribution of the number of requests arriving within the first 5 seconds. (c) the distribution of the arrival time of the nth request; give its mean and its variance. (d) the approximate probability that more than 7250 requests arrive within the first hour.

6. Arrivals of the Number 2 bus form a Poisson process of rate 2 per hour, and arrivals of the Number 7 bus form a Poisson process of rate 7 buses per hour, independently.

(a) What is the probability that exactly three buses pass by in an hour?

(b) What is the probability that exactly three number 7 buses pass by while I am waiting for a number 2 bus?

(c) When the maintenance depot goes on strike, each bus breaks down independently with probability half before reaching my stop. In that case, what is the probability that I wait for 30 minutes without seeing a single bus?

7. Let N_t be a Poisson process of rate λ. Define $X_n = N_n - n$ for $n = 0, 1, 2, \ldots$.

   Explain why X_n is a Markov chain and give its transition probabilities.

   Using Stirling's formula or otherwise, show that the chain is recurrent if and only if $\lambda = 1$.

   If $\lambda = 1$, is it null recurrent or positive recurrent?

8. Members arrive at a snooker club as a Poisson process of rate 1 every 10 minutes after it opens at 6pm. If two members meet, they will play a match together and then leave; but the members are impatient, and anyone who does not get an opponent within 10 minutes of arriving will give up and spend the evening in the pub.

   (a) Let p_n be the probability that the nth member to arrive gets a game. Find p_n for (i) $n = 1$; (ii) $n = 2$; (iii) general n. How does p_n behave as n becomes large? [*Hint: you could also consider the probability that the nth member to arrive finds an opponent already waiting for him.*]

   (b) Find the probability that the first 6 members to arrive all get a game.

   (c) Find the probability that no match starts before 6.20pm.

Additional problems:

9. Passengers arrive at a bus stop at rate 1 per minute. Find the distribution of the number of passengers boarding a typical bus in two cases: (a) buses arrive regularly every 10 minutes; (b) buses arrive as a Poisson process with rate 1 per 10 minutes. Which one has higher variance?

   I arrive at the bus stop at 2pm. Find the distribution of the number of other passengers boarding the same bus as me in the two cases above.

10. Let N_t be a Poisson process of rate λ. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with mean μ and variance σ^2, independent of the process N. Define

$$S_t = \sum_{n=1}^{N_t} Y_n.$$

   The process S_t is called a *compound Poisson process*; each point of the Poisson process N is assigned a weight, and the value of S_t is the sum of the weights of all the points in $[0, t]$.

   Show that $t^{-1/2}\left(S_t - \lambda \mu t\right)$ converges in distribution as $t \to \infty$, and find the limit.