Question 8

Members arrive at a snooker club as a Poisson process of rate 1 per every 10 minutes after it opens at 6pm. If two members meet, they will play a match together and then leave; but the members are impatient, and anyone who does not get an opponent within 10 minutes of arriving will give up and spend the evening in the pub.

(a) Let p_n be the probability that the nth member to arrive gets a game. Find p_n for (i) $n = 1$; (ii) $n = 2$; (iii) general n. How does p_n behave as n becomes large? [*Hint: you could also consider the probability that the nth member to arrive finds an opponent already waiting for him.*]

(b) Find the probability that the first 6 members to arrive all get a game.

(c) Find the probability that no match starts before 6.20pm.

Lemma. *Say $a, b \in \mathbb{R}$ with $a \neq 0$ and $a \neq 1$. Then the recurrence $f_{n+1} = af_n + b$ has the solution*

$$f_n = \left(\frac{f_1}{a} - \frac{b}{a(1-a)}\right) a^n + \frac{b}{1-a}$$

Proof. The general solution to the homogeneous equation $f_{n+1} = af_n$ is $f_n = Ca^n$ for C constant. Noting that $f_n = \frac{b}{1-a}$ is a particular solution to the original recurrence, the general solution is $f_n = Ca^n + \frac{b}{1-a}$. Putting $n = 1$ gives $f_1 = Ca + \frac{b}{1-a}$ and we obtain $C = \frac{f_1}{a} - \frac{b}{a(1-a)}$. □

We shall number the members of the snooker club using the order in which they arrive. Also, we write $\lambda = 10^{-1}$ for their arrival rate.

(a) Let Q_n denote the event that the nth member finds an opponent already waiting for him, and let $q_n = \mathbb{P}(Q_n)$. Clearly $q_1 = 0$ and in general Q_{n+1} occurs if and only if the nth player didn't find a player already waiting *and* he is still waiting when the next player arrives. Therefore

$$\begin{aligned}
q_{n+1} &= \mathbb{P}(\text{player } n \text{ doesn't find a player waiting and is still waiting when player } n+1 \text{ arrives}) \\
&= \mathbb{P}(Q_n^C)\mathbb{P}(\text{player } n \text{ is waiting when player } n+1 \text{ arrives} | Q_n^C) \\
&= (1 - q_n)\mathbb{P}(Y_{n+1} \leq 10) \\
&= (1 - q_n)(1 - e^{-1}) \\
&= (e^{-1} - 1)q_n + (1 - e^{-1})
\end{aligned}$$

having used that the interarrival time Y_{n+1} is an $\mathrm{Exp}(\frac{1}{10})$ random variable with c.d.f. $x \mapsto 1 - e^{-x/10}$. Applying the above Lemma with $a = e^{-1} - 1$ and $b = 1 - e^{-1}$ we get that for $n \geq 1$,

$$q_n = \left(\frac{q_1}{e^{-1} - 1} - \frac{1 - e^{-1}}{(e^{-1} - 1)(2 - e^{-1})}\right)(e^{-1} - 1)^n + \frac{1 - e^{-1}}{2 - e^{-1}} = \frac{(e^{-1} - 1)^n}{2 - e^{-1}} + \frac{1 - e^{-1}}{2 - e^{-1}}$$

The nth player gets a game if the previous member is still waiting for him or if the next one arrives within 10 minutes. Noting that occurrence of the event Q_n is determined by the first n interarrival times, we see that Q_n and $\{Y_{n+1} \leq 10\}$ are independent events and therefore

$$\begin{aligned}
p_n &= \mathbb{P}(Q_n \cup \{Y_{n+1} \leq 10\}) \\
&= \mathbb{P}(Q_n) + \mathbb{P}(Y_{n+1} \leq 10) - \mathbb{P}(Q_n \cap \{Y_{n+1} \leq 10\}) \\
&= q_n + (1 - e^{-1}) - q_n(1 - e^{-1}) \\
&= (1 - e^{-1}) + e^{-1}q_n \\
&= (1 - e^{-1}) + e^{-1}\left(\frac{(e^{-1} - 1)^n}{2 - e^{-1}} + \frac{1 - e^{-1}}{2 - e^{-1}}\right) \\
&= \frac{1}{2e - 1}(e^{-1} - 1)^n + \frac{(1 - e^{-1})(2e - 1) + (1 - e^{-1})}{2e - 1} \\
&= \frac{1}{2e - 1}(e^{-1} - 1)^n + 1 - \frac{1}{2e - 1} \\
&\to 1 - \frac{1}{2e - 1} \approx 0{,}77 \text{ as } n \to \infty
\end{aligned}$$

with the limit existing since $|e^{-1}-1|<1$. As special cases we get

$$p_1 = \frac{1}{2e-1}(e^{-1}-1)^1 + 1 - \frac{1}{2e-1} = \frac{e^{-1}-2}{2e-1} + 1 = 1 - e^{-1} \approx 0,63$$
$$p_2 = \frac{1}{2e-1}(e^{-1}-1)^2 + 1 - \frac{1}{2e-1} = \frac{e^{-2}-2e^{-1}+1-1}{2e-1} + 1 = 1 - e^{-2} \approx 0,86$$

(b) Note that the first 6 members all get a game if and only if the second, fourth and sixth interarrival time are all at most 10 minutes. Using their independence,

$$\mathbb{P}(\{Y_2 \le 10\} \cap \{Y_4 \le 10\} \cap \{Y_6 \le 10\}) = \mathbb{P}(Y_2 \le 10)\mathbb{P}(Y_4 \le 10)\mathbb{P}(Y_6 \le 10) = (1-e^{-1})^3 \approx 0,25$$

(c) Observe that a match takes place before 6.20pm if and only if the first and the second player or the second and the third player meet before 6.20pm. Since these events are disjoint, this has probability

$$\mathbb{P}(\text{players 1, 2 matched before 6.20pm}) + \mathbb{P}(\text{players 2, 3 matched before 6.20pm})$$

Players 1 and 2 are matched before 6.20pm if and only if the second player arrives before 6.20pm and within 10 minutes of the first player arriving. Expressing this in terms of the interarrival times Y_1 and Y_2 and using their independence we get

$$\begin{aligned}
\mathbb{P}(\text{players 1, 2 matched before 6.20pm}) &= \mathbb{P}(Y_1 + Y_2 < 20, Y_2 \le 10) \\
&= \int_{b=0}^{10}\int_{a=0}^{20-b} p_{Y_1,Y_2}(a,b)\,\mathrm{d}a\,\mathrm{d}b \\
&= \int_{b=0}^{10}\int_{a=0}^{20-b} p_{Y_1}(a)p_{Y_2}(b)\,\mathrm{d}a\,\mathrm{d}b \\
&= \int_{b=0}^{10}\int_{a=0}^{20-b} \lambda e^{-\lambda a}\lambda e^{-\lambda b}\,\mathrm{d}a\,\mathrm{d}b \\
&= \lambda\int_{b=0}^{10} e^{-\lambda b}(1-e^{-\lambda(20-b)})\,\mathrm{d}b \\
&= 1 - e^{-10\lambda} - 10\lambda e^{-20\lambda} = 1 - e^{-1} - e^{-2}
\end{aligned}$$

Players 2 and 3 are matched before 6.20pm if and only if the third player arrives before 6.20pm (i.e. $Y_1 + Y_2 + Y_3 < 20$) and the second player *doesn't* arrive within 10 minutes of the first one ($Y_2 > 10$), so that they are not matched. Note that these two conditions entail that $Y_3 \le 10$. Hence

$$\begin{aligned}
\mathbb{P}(\text{players 2, 3 matched before 6.20pm}) &= \mathbb{P}(Y_1 + Y_2 + Y_3 < 20, Y_2 > 10) \\
&= \int_{b=10}^{20}\int_{a=0}^{20-b}\int_{c=0}^{20-a-b} p_{Y_1,Y_2,Y_3}(a,b,c)\,\mathrm{d}c\,\mathrm{d}a\,\mathrm{d}b \\
&= \int_{b=10}^{20}\int_{a=0}^{20-b}\int_{c=0}^{20-a-b} \lambda e^{-\lambda a}\lambda e^{-\lambda b}\lambda e^{-\lambda c}\,\mathrm{d}c\,\mathrm{d}a\,\mathrm{d}b \\
&= \int_{b=10}^{20} \lambda e^{-\lambda b}\int_{a=0}^{20-b} \lambda e^{-\lambda a}(1-e^{-\lambda(20-a-b)})\,\mathrm{d}a\,\mathrm{d}b \\
&= \int_{b=10}^{20} \lambda e^{-\lambda b}\left(1 - e^{-\lambda(20-b)} - (20-b)\lambda e^{-\lambda(20-b)}\right)\mathrm{d}b \\
&= (e^{-10\lambda} - e^{-20\lambda}) - \lambda\int_{b=10}^{20} e^{-20\lambda} + (20-b)\lambda e^{-20\lambda}\,\mathrm{d}b \\
&= e^{-10\lambda} - e^{-20\lambda} - 10\lambda e^{-20\lambda} - 200\lambda^2 e^{-20\lambda} + \lambda^2 e^{-20\lambda}\frac{20^2-10^2}{2} \\
&= e^{-10\lambda} - e^{-20\lambda} - 10\lambda e^{-20\lambda} - 50\lambda^2 e^{-20\lambda} \\
&= e^{-1} - e^{-2} - e^{-2} - \frac{1}{2}e^{-2} = e^{-1} - \frac{5}{2}e^{-2}
\end{aligned}$$

So finally the probability that a match takes place before 6.20pm is

$$\mathbb{P}(\text{match before 6.20pm}) = (1 - e^{-1} - e^{-2}) + (e^{-1} - \frac{5}{2}e^{-2}) = 1 - \frac{7}{2}e^{-2} \approx 0,53$$