Remarks on automorphism groups of groups

Minhyong Kim

In GAGA sheet 5 (2013, HT), the notion of the automorphism group

$$\mathrm{Aut}(G)$$

of a group G was studied, about which it might be good to add a few comments.

First, let's recall the automorphisms $\mathrm{Aut}(X)$ of an arbitrary set X, not necessarily a group. These are just the invertible set maps from X to X. I think you know the fact that $\mathrm{Aut}(X)$ is always a group. This is because composition of maps always defines a group law, regardless of what structure we have or lack on X. When $X = \{1, 2, \ldots, n\}$, for example, we just get the symmetric group S_n.

However, when X is a group, which we therefore call G, we can consider two different objects, the automorphisms of G regarded merely as a set, or the automorphisms of G as a group. To distinguish the two, they might be denoted

$$\mathrm{Aut}_{set}(G)$$

and

$$\mathrm{Aut}_{gp}(G),$$

respectively. The definition of the group structure is always given by composition, so it's clear that the latter is a *subgroup* of the former. That is, they are the set automorphisms that respect the group structure on G.

Since we have set the context, we will now drop the subscript and refer to $\mathrm{Aut}_{gp}(G)$ as simply $\mathrm{Aut}(G)$. If you are given a group G, it is immediately an interesting problem to compute $\mathrm{Aut}(G)$. For example,

$$\mathrm{Aut}(\mathbb{Z}) \simeq C_2.$$

This already shows the restrictions placed on a set automorphism by the requirement of preserving the group structure. You might be able to see that $\mathrm{Aut}_{set}(\mathbb{Z})$ is enormous, in fact, uncountable.

Smaller versions of this phenomenon can be seen from the isomorphism

$$\mathrm{Aut}(C_n) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}.$$

What is $\mathrm{Aut}_{set}(C_n)$?

Here are some more computations to verify:

$$\mathrm{Aut}(\mathbb{Q}) \simeq \mathbb{Q}^{\times},$$

$$\mathrm{Aut}(C_2 \times C_2) \simeq GL_2(\mathbb{F}_2),$$

the invertible 2×2 matrices over the finite field $\mathbb{F}_2$. The reason for this is that

$$C_2 \times C_2 \simeq \mathbb{F}_2^2,$$

which can be regarded as a two-dimensional vector space over $\mathbb{F}_2$. But for $\mathbb{F}_2$-vector spaces, group homomorphisms are the same as vector space homomorphisms (that is, linear maps). Here is a general fact: Suppose V is a vector space over K, where $K = \mathbb{Q}$ or $K = \mathbb{F}_p$. Then

$$\mathrm{Aut}(V) = \mathrm{Aut}_K(V).$$

Here, $\mathrm{Aut}(V)$ just refers to the automorphisms of V regarded as a group under vector space addition, while $\mathrm{Aut}_K(V)$ refers to the invertible K-linear maps.

However, this will not be true for vector spaces over other fields. For example, if we regard $\mathbb{R}$ as a vector space over itself, then

$$\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}) = \mathbb{R}^*,$$

but

$$\mathrm{Aut}(\mathbb{R})$$

is something quite hard to describe. Obviously, we will always have

$$\mathrm{Aut}_K(V) \subset \mathrm{Aut}(V).$$

Incidentally, for the $C_2 \times C_2$ case, you might enjoy figuring out the isomorphism

$$GL_2(\mathbb{F}_2) \simeq S_3.$$

Hint: When someone asks you to define a homomorphism

$$G \longrightarrow S_n,$$

(which might be an isomorphism) you should ask yourself: 'How can I make G act on a set with n elements?'

In considering the problem of computing $\mathrm{Aut}(G)$, you should first ask yourself, 'How can I produce any non-trivial homomorphism at all from G to G?' Later on, you will encounter such a question for much more general mathematical structures S, that is, the question of producing automorphisms of S, or computing $\mathrm{Aut}(S)$ (where these will be invertible maps preserving whatever structure is under discussion). In the case where G is a group, we get some automorphisms for free. These are the automorphisms

$$\Theta_x : G \simeq G$$

that you get from elements x of the group G itself. Recall the definition from sheet 5:

$$\Theta_x(g) = xgx^{-1}.$$

There automorphisms are also called the *inner automorphisms* of G.

However, we should be careful that an inner automorphism might be trivial. That is, conjugation by x might do nothing to the elements of G. For example, G is an abelian group if and only if all inner automorphisms are trivial. In this case, the inner automorphisms give no information at all about the automorphism group. For example, none of the automorphisms in

$$\mathrm{Aut}(C_2 \times C_2) \simeq GL_2(\mathbb{F}_2)$$

are inner. On the other hand, it can happen that *all* automorphisms are inner, such as for S_3, also described in sheet 5. In fact, that sheet asks you to show that via the map

$$x \mapsto \Theta_x,$$

we get an isomorphism

$$S_3 \simeq \mathrm{Aut}(S_3).$$

With a bit more work, you can show

$$S_n \simeq \mathrm{Aut}(S_n)$$

for all $n \neq 2, 6$. I hope it's obvious what happens for $n = 2$. In the case of $n = 6$, it's quite an interesting exercise to try to locate an automorphism of S_6 that is not inner.

A question raised by Sam during the tutorial is quite amusing. Since $\mathrm{Aut}(G)$ is itself a group, we can ask about

$$\mathrm{Aut}(\mathrm{Aut}(G)),$$

ot indeed, iterate the construction any number of times by recursively defining

$$\mathrm{Aut}^n(G) := \mathrm{Aut}(\mathrm{Aut}^{n-1}(G)).$$

What kind of patterns emerge here?

Let's try just a few simple examples. For $G = C_5$, we get

$$\mathrm{Aut}(C_5) = \mathbb{F}_5^* \simeq C_4.$$

$$\mathrm{Aut}^2(C_5) = \mathrm{Aut}(C_4) \simeq (\mathbb{Z}/4\mathbb{Z})^* \simeq C_2.$$

$$\mathrm{Aut}^3(C_5) = \mathrm{Aut}(C_2) = 1.$$

Hence,

$$\mathrm{Aut}^n(C_5) = 1$$

for all $n \geq 3$.
For $n \neq 2, 6$,

$$\mathrm{Aut}^m(S_n) = S_n$$

for all m.
For C_8, we get

$$\mathrm{Aut}(C_8) \simeq (\mathbb{Z}/8\mathbb{Z})^* \simeq C_2 \times C_2.$$

So

$$\mathrm{Aut}^2(C_8) \simeq \mathrm{Aut}(C_2 \times C_2) \simeq S_3,$$

from which we deduce

$$\mathrm{Aut}^n(C_8) = S_3$$

for all $n \geq 2$.

A rather tricky question might be:

> For which finite groups G, does the sequence of groups $\mathrm{Aut}^n(G)$ stabilize?

Can you think of an example where it does not?