Probability, sheet 4

There was some mystery today about convergence in problem number 10, which I will try to resolve here.

We are asked to prove the identity

$$\frac{\sin t}{t} = \prod_{n=1}^{\infty} \cos(t/2^n)$$

by comparing characteristic functions of random variables. Let X be uniformly distributed on $[-1, 1]$. Then its characteristic function is

$$(1/2)\int_{-1}^{1} e^{itx} dx = [e^{it} - e^{-it}]/(2it) = \sin(t)/t.$$

Now let Y_n be the distribution that takes values 1 and -1 each with probability $1/2$. Then $2^{-n}Y_n$ takes the values 2^{-n} and -2^{-n} with probability $1/2$. So the characteristic function of

$$Z = \sum_{n=1}^{N} 2^{-n} Y_n$$

is

$$\prod_{n=1}^{N} E(e^{it2^{-n}Y_n})$$

$$= \prod_{n=1}^{N} ((1/2)e^{it/2^n} + (1/2)e^{-it/2^n})$$

$$= \prod_{n=1}^{N} \cos(t/2^n).$$

The variable Z will run through 2^{-N} values each with probability 2^{-N}. So a direct calculation of the characteristic function will give

$$S_N = 2^{-N} \sum_{z} e^{itz},$$

where the sum runs over the values of Z. Now here is the key point, which I will leave to you to verify:

S_N is a Riemann sum for the integral

$$\int_{-1}^{1} e^{itx} dx.$$

Thus, as $N \to \infty$, S_N will converge to the integral, which we have already calculated to be $\sin(t)/t$. This gives the desired identity.