Group Theory, HT, 2012, sheet 2, exercise 11

asks the following question: A group G acts faithfully on a set of five elements with two orbits, one of order 3 and the other of order 2. What are the possibilities for G?

Let us denote the set by S. The action corresponds to a homomorphism

$$G \longrightarrow \mathrm{Aut}(S),$$

which is injective since the action is faithful. So up to isomorphism, we need only consider subgroups of $\mathrm{Aut}(S)$. We label the elements of S as

$$S = \{1, 2, 3, 4, 5\},$$

allowing us to identify $\mathrm{Aut}(S)$ with S_5 and G with a subgroup of S_5. We can choose the labelling so that the two orbits are

$$A = \{1, 2, 3\}, \quad B = \{4, 5\}.$$

Identify $S_3 = \mathrm{Aut}(A)$ with the subgroup of S_5 that fixes the two elements $\{4, 5\}$ and $S_2 = \mathrm{Aut}(B)$ with the subgroup fixing $\{1, 2, 3\}$. These two subgroups intersect at the identity and commute with each other, so that

$$S_3S_2 \simeq S_3 \times S_2.$$

Given any element $g \in G$, we can consider the restrictions $g|A$ and $g|B$ defining homomorphisms

$$\rho_1 : G \longrightarrow S_3$$

and

$$\rho_2 : G \longrightarrow S_2.$$

Meanwhile, an element of S_5 is determined by the action on the elements of A and B, so that we have

$$G \subset S_3S_2.$$

Note that ρ_2 must hit the non-trival element (45) since the action is transitive. Similarly, ρ_2 must map G to a transitive subgroup of S_3, that is, A_3 or S_3.

Now check the following:

In the A_3 case, $G = A_3S_2 \simeq A_3 \times S_2$. The key point here is that in this case, one can show $(45) \in G$. This implies the result rather easily.

In the S_3 case, two possibilities occur.

(1) $G = S_3S_2 \simeq S_3 \times S_2$;

(2) G is the subgroup generated by (123) and $(23)(45)$. This subgroup is easily seen to be isomorphic to S_3, for example, by simply writing down the elements. (But it is not *not equal* to the S_3 subgroup under discussion.)

This last case is what I wanted to warn you about. But then, I got confused myself because Dave's explanation was quite convincing. That is, this is the case of an S_3 subgroup that is 'diagonally embedded' in $S_3 \times S_2$, in a way that it surjects onto both components.