Some remarks on symmetry groups

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In this note, we will discuss a bit the basic framework that goes into the computation of the symmetry groups of various objects.

A key fact is:

> Any isometry ϕ of Euclidean n-space $\mathbb{R}^n$ can be uniquely written as
>
> $$\phi = T_v \circ A,$$
>
> where
>
> $$A : \mathbb{R}^n \to \mathbb{R}^n$$
>
> is a linear map defined by an orthogonal matrix, and
>
> $$T_v(x) = x + v$$
>
> is the translation map by a vector v.

If you have seen this fact for $n = 2$ and 3, this is all we will need. It says that the condition that a map preserve distances is extremely rigid. Two predictable ways of constructing such a map, via linear orthogonal transformations and translations, exhaust all possibilities. You may have learned the fact that any orthogonal transformation can be written as a rotation composed with a reflection[1]. Thus, any isometry is a rotation, a reflection, a translation, or a composition of these. As an important consequence, note that if $\phi(0) = 0$, then $v = 0$, so ϕ is just an orthogonal linear transformation. The linearity here is crucial in many applications, as we now explain.

A basic fact is:

> If
>
> $$L, M : V \to W$$
>
> are linear maps between vector spaces and $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for V, then $L(b_i) = M(b_i)$ for every i implies that $L = M$.

This is obvious and elementary, but very important to keep in mind. It is one of the important reasons to keep track of bases, since a linear transformation is completely by its effect on basis elements.

To illustrate its utility, let us consider the simple problem where

$$A : \mathbb{R}^2 \to \mathbb{R}^2$$

is a linear map with ± 1 as eigenvalues, and corresponding eigenvectors v and w which are assumed to be orthogonal. So $Av = v$ and $Aw = -w$. We would like to conclude that A is equal to the reflection r_l across the line l spanned by v. How do we legitimately reach this conclusion? Well,

$$Av = v = r_l(v)$$

and

$$Aw = -w = r_l(w),$$

and $\{v, w\}$ is a basis. (Why?) Therefore, $r_l = A$.

[1] This is easy to see in two dimensions, somewhat harder in three dimensions. Search on the Bloomsbury Journal website for the entry 'Isometries in two and three dimensions.'

Now consider the problem of determining the group G_L of isometries that stabilise a line L in the plane. Let us compute G_L rather systematically: Let $\phi \in G_L$. Choose any $O \in L$, and compute $P = \phi(O) \in L$. Then the vector v pointing from P to O is parallel to L, and hence $T_v(L) = L$. Thus,

$$T_v \circ \phi(L) = L,$$

that is, $\psi = T_v \circ \phi$ is an isometry of the plane that stabilizes L, but furthermore, that sends O to itself. Hence, taking O as the origin, we can then assume that ψ is an orthogonal matrix stabilising the line L that goes through the origin. Consider now the possibilities for ψ. Let b be a vector lying in L. Then $\psi(b)$ still lies in L, but must have the same length as b. Therefore,

$$\psi(b) = \pm b.$$

Now let c be a vector orthogonal to b. Since ψ preserves lengths *and* angles, we see that $\psi(c) = \pm c$ as well.

So here are the possibilities.

(1)

$$\psi(b) = b \ \ ; \psi(c) = c.$$

Then $\psi = I$.

(2)

$$\psi(b) = b \ \ ; \psi(c) = -c.$$

Then ψ is the reflection across the line L.

(3)

$$\psi(b) = -b \ \ ; \psi(c) = c.$$

Then ψ is the reflection across the line $L^{\perp}$ orthogonal to L that goes through O.

(4)

$$\psi(b) = -b \ \ ; \psi(c) = -c.$$

Then ψ is the rotation operator through an angle π around O.

In each case, how do we know these are the correct linear maps? This is again because ψ and the linear maps described *have the same effect on the basis* $\{b, c\}$. To check this efficiently, we made key use of the fact that the maps under consideration are linear.

To conclude then,

$$\phi = T_{-v} \circ \psi$$

is one of these four maps followed by translation by a vector $-v$ parallel to L. That is, such a ϕ clearly lies in G_L. We have just shown that all elements of G_L have this form.

Now we consider the symmetries of the cube. First, let us say clearly what is meant. Let $\mathcal{C} \subset \mathbb{R}^3$ be a cube. We want to know the group $G_{\mathcal{C}}$ of all isometries ϕ of $\mathbb{R}^3$ such that $\phi(\mathcal{C}) = \mathcal{C}$. It would be useful to reduce again to the case of orthogonal transformations. To do this, note that ϕ must take the centre of the cube to itself. (Why?) Now we can construct a new orthogonal coordinate system for $\mathbb{R}^3$ so that $\mathcal{C}$ has its centre at the origin. (You may want to make this part of the argument more rigorous.) Thus, we can regard the symmetries as being a subgroup of the group of orthogonal matrices. Next, notice that vectors pointing to any three of the vertices of the cube are linearly independent. (You should prove for yourself that no three lie on a plane through the origin.) Thus any element of $G_{\mathcal{C}}$ is completely determined by its effect on any three of the vertices. Let us fix one vertex and label it A, assuming that it lies above the $x - y$ plane, for convenience of reference. Consider the subgroup H of $G_{\mathcal{C}}$ generated by a rotation R by $\pi/2$ around the z axis, and the reflection r across the $x - y$ plane. It is easy to prove that

$$H = \{I, R, R^2, R^3, r, rR, rR^2, rR^3\}.$$

To see this, it suffice to show that H is indeed a subgroup. The containment of the identity and inverses is rather obvious, so one needs to show that this set is closed under composition of linear maps. For this, the key thing to check is that

$$Rr = rR.$$

(Fill in the other details.) How do we check this? It suffices to show that these two transformations have the same effect on three of the vertices, which is easy. Next, you should check that H has order 8, that is, that the elements are distinct. It is clear that the R^i are distinct, and therefore, so are the rR^i. (Check the effect on A.) We see that the R^i are distinct from the rR^j once again by the effect on A, since the latter ones will send A to a point below the $x - y$ plane. Meanwhile, H clearly acts transitively on the 8 vertices, and hence, it acts simply transitively. Therefore, for any vertex v, there is a unique element $h \in H$ such that $h(v) = A$. Why have we bothered to prove this? The point is that now, for any $\phi \in G_{\mathcal{C}}$, there is a unique h such that $\phi = h \circ \psi$, where $\psi \in K$, the stabiliser of A in $G_{\mathcal{C}}$. What is this stabiliser? Note that A has three neighbouring vertices on the cube, say B, C, D. These must be taken to each other by any $\psi \in K$, since the distance to A must remain the same after application. Therefore, the plane spanned by these three vertices is stabilised by ψ. So we get a homomorphism

$$K \longrightarrow T,$$

where T consists of the isometries of the plane that stabilise the triangle spanned by B, C, D. This map is injective, because the vectors corresponding to B, C, D in three space are linearly independent. I hope you have already proved that T is isomorphic to S_3, generated by a rotation through the angle $2\pi/3$ and a reflection. I also leave it to you to prove that K also surjects onto T, simply by constructing the rotation (around a long diagonal going through A) and the reflection (across a the plane that contains this long diagonal and the point B) of three space that maps to the generators of T. (You should convince yourself that both of these lie in K). Therefore,

$$K \simeq T,$$

and K is also generated exactly by this rotation and reflection. So now, we have described completely the stabiliser K, and we know that any element of $G_{\mathcal{C}}$ can be uniquely written as a product of an element of H and an element of K. In particular, $|G_{\mathcal{C}}| = 48$.

I encourage you now to work out the symmetries of the prism in the same spirit.

A small word of warning: You will note that the proof given above also could be made more precise, for example, regarding the coordinate systems and verification of the effect of some transformations on vertices. I made also some appeal to visual arguments. Now, it is true that to write down all possible details in a discussion like this can get pretty tedious. In an exam, on the other hand, your goal is to put in just the right amount of detail to display a balanced understanding of the material. Such a balanced understanding includes a good awareness of what details would need to be added, were a skeptical reader to ask for them.